Chapter 4. Heighest Weight Modules I.

4.1 Simple submodules of Verma modules

Prop For any $\lambda\in\mathfrak{h}^*$, the module $M(\lambda)$ has a unique simple socle.

Pf. Assume L and L' are two simple submods of $M(\lambda)$.

Then identifying $M(\lambda)$ with $\mathcal{U}(\mathfrak{n}^-)$ as $\mathcal{U}(\mathfrak{n}^-)$-mod, L and L' are two left ideals of $\mathcal{U}(\mathfrak{n}^-)$.

Claim: $L\cap L'\neq 0$ (then our prop follows), it follows from the Lemma immediately:

Lemma: Given left noetherian R with no zero-divisors, then any two nonzero left ideals of R intersect nontrivially.

Pf: Let I, J be two nonzero ideals of R with $I\cap J=0$.

Take $0\neq x\in J$. Consider the sequence

$$0\subset I\subset I+Ix\subset I+Ix+Ix^2\subset\cdots\cdots$$

We claim that they're direct sums: Let $a_0+a_1x+\cdots+a_nx^n=0$, $a_i\in I$.

Then $a_0=(a_1+a_2x+\cdots+a_nx^{n-1})x\in J \Rightarrow a_0=0$

Since R has no zero-divisors, $a_1+\cdots+a_nx^{n-1}=0 \rightsquigarrow a_1=0\rightsquigarrow\cdots$

Thus $a_i=0$ $\forall i$. which means each sum is direct.

This contradicts with the noetherianity of R. The proof completes.

Exercise: Let M be a nonzero submod of $M(\lambda)$ with a nondegenerated contrav. form $(\cdot,\cdot)$, prove that M is the simple socle of $M(\lambda)$.

Pf: It suffices to show that M is simple.

Assume $L(\mu)\subseteq M$ be the simple submodule. Then by Thm 3.15, the restriction of $(\cdot,\cdot)_M$ on $L(\mu)$ is also nondegenerated. Then $L(\mu)^\perp=0$.

(Otherwise $L(\mu)^\perp \supset L(\mu)$) Thus $\Pi(L(\mu)) = \Pi(M)$, (the weight set)
Now consider each (fin dim) weight space: if $v \in M_\gamma \setminus L(\mu)_\gamma$, $\exists w \in L(\mu)_\gamma$ s.t.
$(v, \bullet)|_{L(\mu)_\gamma} = (w, \bullet)|_{L(\mu)_\gamma}$, Then $v - w \in L(\mu)^\perp$, i.e. $v = w$.
Therefore, $M = L(\mu)$ is simple.

4.2. Homom btw Verma modules.

<u>Thm</u>. Let $\lambda, \mu \in \mathfrak{h}^*$,
(a) Any nonzero homom $\varphi : M(\lambda) \to M(\mu)$ is injective.
(b) $\dim \mathrm{Hom}(M(\mu), M(\lambda)) \leq 1$
(c) The unique socle $L(\mu)$ in $M(\lambda)$ is itself a Verma module.

Pf: <u>a)</u> Let $\varphi(v_\lambda^+) = u v_\mu^+$ for some $u \in U(\mathfrak{n}^-) \setminus \{0\}$.
Assume $\varphi(u' v_\lambda^+) = u' \varphi(v_\lambda^+) = u' u \, v_\mu^+ = 0$, Then $u'u = 0$.
Since $U(\mathfrak{n}^-)$ has no zero-divisors, $u' = 0$, i.e. $u' v_\lambda^+ = 0$.

<u>b)</u> Let $\varphi_1, \varphi_2 \in \mathrm{Hom}(M(\mu), M(\lambda)) \setminus \{0\}$ and S_μ, S_λ be simple socles of $M(\mu)$ and $M(\lambda)$. By a), we view φ_i as embeddings.
By Schur's lemma, $\exists c \in \mathbb{C}$ s.t. $\varphi_1 - c\varphi_2 |_{S_\mu} = 0$.
By injectivity of $\varphi_1 - c\varphi_2$, it must be zero.

<u>c)</u> $M(\mu) \overset{\pi}{\twoheadrightarrow} L(\mu) \overset{i}{\hookrightarrow} M(\lambda)$, $i \circ \pi : M(\mu) \to M(\lambda)$ injective
$\Rightarrow \pi$ is also injective. Then $M(\mu) = L(\mu)$.

4.3 Dominant Integral Weights

Dominant: $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{\leq 0}$, $\forall \alpha \in \Phi^+$
Integral: $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$, $\forall \alpha \in \Phi^+$
regular: $|W \cdot \lambda| = |W|$

<u>Lemma</u>: $\lambda \in \mathfrak{h}^*$ regular iff $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$, $\forall \alpha \in \Phi$

pf: "$\Rightarrow$" $s_\alpha \cdot \lambda = s_\alpha(\lambda+\rho)-\rho = \lambda - \langle \lambda+\rho, \alpha^\vee \rangle \alpha \neq \lambda \Rightarrow \langle \lambda+\rho, \alpha^\vee \rangle \neq 0$.

"$\Leftarrow$" Take $\Delta = \Delta(\lambda+\rho) = \{\alpha \in \Phi : \langle \lambda+\rho, \alpha^\vee \rangle > 0\}$

Suppose $w \cdot \lambda = \lambda$ i.e. $w(\lambda+\rho) = \lambda+\rho$. with $w \neq 1$.

Then w^{-1} maps a positive root β into a negative root (w.r.t Δ)

$(\lambda+\rho, \beta) = (w(\lambda+\rho), \beta) = (\lambda+\rho, w^{-1}(\beta)) < 0$, which is a confliction

Rmk. $\lambda \in \mathfrak{h}^*$ dominant, integral and regular $\Longleftrightarrow$ $\lambda \in \Lambda^+ \rightsquigarrow \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0}, \forall \alpha \in \Phi^+$

pf. A fact: If Δ is the set of simple roots, then $\Delta^\vee$ is the set of simple roots of $\Phi^\vee$

"$\Rightarrow$" dominant & integral : $\langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0}$, $\forall \alpha \in \Phi^+$.

and regular $\Rightarrow$ $\langle \lambda, \alpha^\vee \rangle + \langle \rho, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 1}$, $\forall \alpha \in \Phi^+$

$\Rightarrow \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0}$ $\forall \alpha \in \Delta$ $\Rightarrow \lambda \in \Lambda^+$

"$\Leftarrow$" Obvious.

Prop. For a dominant and integral weight $\lambda \in \mathfrak{h}^*$, $w \in W$ with a reduced expression $w = s_n \cdots s_1$, we set $\lambda_i = (s_i \cdots s_1) \cdot \lambda$ $\forall i = 0, \cdots, n$. ($\lambda_0 = \lambda$). Then we have a seq of embeddings:

$$M(w \cdot \lambda) = M(\lambda_n) \subset M(\lambda_{n-1}) \subset \cdots \subset M(\lambda_1) \subset M(\lambda_0) = M(\lambda)$$

Moreover, if λ is also regular, these embeddings are strict.

pf. λ: $\langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0}$ for any $\alpha \in \Delta \Longleftrightarrow \langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0}$, $\forall \alpha \in \Phi^+$

It suffices to show that $\lambda_{i+1} \leq \lambda_i$ $\forall i$, that is, $\langle \lambda_i+\rho, \alpha_{i+1}^\vee \rangle \geq 0$,

since $\lambda_{i+1} = (s_{i+1} \cdots s_1) \cdot \lambda = s_{i+1} \cdot \lambda_i = s_{i+1}(\lambda_i+\rho) - \rho = \lambda_i - \langle \lambda_i+\rho, \alpha_{i+1}^\vee \rangle \alpha_{i+1}$.

$$\langle \lambda_i+\rho, \alpha_{i+1}^\vee \rangle = \langle (s_i \cdots s_1) \cdot \lambda + \rho, \alpha_{i+1}^\vee \rangle = \langle (s_i \cdots s_1)(\lambda+\rho), \alpha_{i+1}^\vee \rangle$$
$$= \langle \lambda+\rho, (s_1 \cdots s_i)(\alpha_{i+1}^\vee) \rangle \geq 0$$

(By lemma 10.2 C [GTM9], if $s_1 \cdots s_t$ is reduced, $(s_1 \cdots s_{t-1})(\alpha_t) > 0$)

Moreover, if λ is regular, $\langle \lambda+\rho, (s_1 \cdots s_i)(\alpha_{i+1}^\vee) \rangle \neq 0$. Thus the last statement follows.

Ex. a) Assume $\lambda+\rho \in \Lambda^+$, then the unique socle of $M(\lambda)$ is $M(w_0\cdot\lambda)$, where w_0 is the longest element in W.

Pf. Firstly, we show that $w_0\cdot\lambda$ is an anti-dominant weight.

$\langle w_0\cdot\lambda+\rho, \alpha^\vee\rangle = \langle w_0(\lambda+\rho), \alpha^\vee\rangle = \langle \lambda+\rho, w_0(\alpha^\vee)\rangle < 0$, $\forall \alpha\in\Phi^+$. $(w_0: \Phi^+ \to \Phi^-)$

Then we show that $M(w_0\cdot\lambda)$ is simple.

Suppose N is a submodule of $M(w_0\cdot\lambda)$. Then N has a maximal weight, say μ. Take a maximal vector v of wt μ.

Then $U(\mathfrak{g})v \simeq M(\mu) \hookrightarrow M(w_0\cdot\lambda)$, which is a contradiction.

Rmk. Prove w_0 is the unique element mapping Φ^+ onto Φ^-.

Pf. Since $l(w) = n(w) = \#\{\alpha\in\Phi^+ : w(\alpha)\in\Phi^-\}$, it suffices to show for each two positive systems Φ_1^+, Φ_2^+, there exists a unique $w\in W$, s.t. $w: \Phi_1^+ \to \Phi_2^+$.

We prove by induction on $|\Phi_1^+ \cap \Phi_2^-|$. If $|\Phi_1^+\cap\Phi_2^-| = 0$, which means $\Phi_1^+ = \Phi_2^+$, then only 1 satisfies the condition $(l(w) = n(w) = 0)$

If $|\Phi_1^+\cap\Phi_2^-| = r > 0$, there exists an $\alpha\in\Delta_1$ s.t. $\alpha\in\Phi_2^-$. Then

$|s_\alpha\Phi_1^+ \cap \Phi_2^-| = r-1$ $(s_\alpha: \Phi_1^+\setminus\{\alpha\} \to \Phi_1^+\setminus\{\alpha\})$

By induction hypothesis, $\exists$ $w: s_\alpha\Phi_1^+ \to \Phi_2^-$. Then $ws_\alpha: \Phi_1^+ \to \Phi_2^-$.

The uniqueness follows from the uniqueness of $1: \Phi_1^+ \to \Phi_1^+$

4.4 Simplicity Criterion (Integral)

Our discussion strongly will rely on Prop 1.4:

Given $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$, Suppose $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^{\geq -1}$ then

$S_\alpha \cdot \lambda = \lambda - (n+1)\alpha = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha$ and $M(S_\alpha \cdot \lambda) \subseteq M(\lambda)$

In particular, if $\lambda \in \Lambda$, $\alpha \in \Delta$ with $S_\alpha \cdot \lambda < \lambda$, i.e. $n \geq 0$, then $M(S_\alpha \cdot \lambda) \subsetneq M(\lambda)$.

<u>Thm</u> Let $\lambda \in \Lambda$. Then $M(\lambda) = L(\lambda)$ iff λ is anti-dominant.

Pf: "$\Rightarrow$" Suppose λ is NOT antidominant, i.e. $\exists \alpha \in \Delta$ s.t. $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$

Then by Prop 1.4, $M(S_\alpha \cdot \lambda) \subsetneq M(\lambda) = L(\lambda)$ which is a confliction.

"$\Leftarrow$" λ is antidominant $\Leftrightarrow$ $M(\lambda)$ has the only factor $L(\lambda)$.

Moreover, $\mathrm{Ext}(L(\lambda), L(\lambda)) = 0$, $M(\lambda) = L(\lambda)$.

<u>Rmk</u>. Proof of "$\Leftarrow$" doesn't require integrality, but "$\Rightarrow$" does!

<u>Ex</u>. If $\lambda \in \Lambda$ is antidominant, then the socle of $P(w\cdot\lambda)$ with $w \in W$ is a direct sum of $L(\lambda)$.

Pf: $P(w\cdot\lambda)$ has a std fol with factors $M(\mu)$, $\mu \in W_{[\lambda]}\cdot\lambda$.

By Prop 1.4, $L(\lambda)$ is the socle of any $M(\mu)$ and $\mathrm{Ext}(L(\lambda), L(\lambda)) = 0$

Thus, $\mathrm{soc}(P(w\cdot\lambda)) = \bigoplus L(\lambda)$.

4.5 & 4.6. Existence of Embeddings (Integral)

Goal: We want to prove if $\lambda \in \Lambda$ with $S_\alpha \cdot \lambda \leq \lambda$ for some $\alpha \in \Phi^+$, then there exists an embedding: $M(S_\alpha \cdot \lambda) \subseteq M(\lambda)$

<u>Lemma</u>. Let $\mathfrak{a}$ is a nilpotent Lie alg, with $x \in \mathfrak{a}$ and $u \in U(\mathfrak{a})$. Given $n \in \mathbb{Z}^{>0}$, there exists an integer t depending on x and u st. $x^t u \in U(\mathfrak{a}) x^n$.

Pf: Let $l_x : \mathfrak{a} \to \mathfrak{a}$; $a \mapsto xa$, $r_x : \mathfrak{a} \to \mathfrak{a}$; $a \mapsto ax$, $ad_x = l_x - r_x$. (commuting)

$\mathfrak{a}$ is nilpotent means $\exists q \in \mathbb{Z}^{>0}$ s.t. $(ad_x)^q = 0$.

$$x^t u = (l_x)^t u = (ad_x + r_x)^t u = \sum_{i=0}^{t} \binom{t}{i} (ad_x^i(u)) x^{t-i}$$

Take $t > q+n$ as required.

Prop. Let $\lambda, \mu \in \mathfrak{h}^*$, $\alpha \in \Delta$ with $n := \langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}$. Assume that

$$M(s_\alpha \cdot \mu) \subseteq M(\mu) \subseteq M(\lambda).$$

Then there are two possibilities for the position of $M(s_\alpha \cdot \lambda)$.

1). If $n \leq 0$, then $\cdots \subseteq M(\lambda) \subseteq M(s_\alpha \cdot \lambda)$

2) If $n > 0$, then
$$\begin{array}{ccccc} M(s_\alpha\cdot\mu) & \subseteq & M(\mu) & \subseteq & M(\lambda) \\ & \rotatebox[origin=c]{-45}{$\subseteq$} & & \rotatebox[origin=c]{45}{$\supseteq$} & \\ & & M(s_\alpha\cdot\lambda) & & \end{array}$$

Pf: 1) follows immediately from Prop 1.4. because $\langle s_\alpha \cdot \lambda + \rho, \alpha^\vee \rangle = -n \geq 0$

2). $M(s_\alpha \cdot \lambda) \subseteq M(\lambda)$ follows immediately from Prop 1.4. Now we consider the remaining embedding:

Since $M(s_\alpha \cdot \mu) \subseteq M(\mu)$, then $\mu - s_\alpha \cdot \mu = \langle \mu+\rho, \alpha^\vee \rangle \alpha \geq 0$.

i.e. $\langle \mu+\rho, \alpha^\vee \rangle \in \mathbb{Z}^{\geq 0}$.

Set $s = \langle \mu+\rho, \alpha^\vee \rangle$. Then $M(s_\alpha \cdot \mu)$ is generated by $y_\alpha^s \cdot v_\mu^+$

Moreover, $M(\mu) \subseteq M(\lambda)$ implies $\exists u \in U(\mathfrak{n}^-)$ s.t. $v_\mu^+ = u v_\lambda^+$.

Then $y_\alpha^s v_\mu^+ = y_\alpha^s u v_\lambda^+$. By lemma, $\exists$ sufficiently large $t \in \mathbb{Z}^{>0}$ s.t.

$y_\alpha^t v_\mu^+ \in U(\mathfrak{n}^-) y_\alpha^n v_\mu^+ \subseteq M(s_\alpha \cdot \lambda)$, since $\mathfrak{n}^-$ is nilpotent.

Now we claim: $y_\alpha^t v_\mu^+ \in M(s_\alpha \cdot \lambda) \overset{t > s}{\Longrightarrow} y_\alpha^{t-1} v_\mu^+ \in M(s_\alpha \cdot \lambda)$.

$\left(x_\alpha y_\alpha^t v_\mu^+ = y_\alpha^t x_\alpha v_\mu^+ + t y_\alpha^{t-1} (h_\alpha - t + 1) v_\mu^+ = t(s-t) y_\alpha^{t-1} v_\mu^+ \in M(s_\alpha \cdot \lambda). \right)$

Thus, $y_\alpha^s v_\mu^+ \in M(s_\alpha \cdot \lambda)$ as desired.

Thm. (Verma) Let $\lambda \in \mathfrak{h}^*$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subseteq M(\lambda)$.

Pf of integral case: ($\lambda \in \Lambda$, then any weights mentioned are integral)

① Since $\mu + \rho \in \Lambda$, there exists $w \in W$ s.t. $w \cdot \mu \in \Lambda^+ - \rho$. Let $w = s_n \cdots s_1$

be a reduced expression of w.

Then by prop 4.3. we have

$$M(\mu') = M(\mu_0) \supseteq M(\mu_1) \supseteq \cdots \supseteq M(\mu_n) = M(\mu)$$

where $\mu' = w^{-1} \cdot \mu$, $\mu_k = (s_k \cdots s_1) \cdot \mu'$ $\quad (\mu_{k+1} = s_{k+1} \cdot \mu_k)$

② Define a parallel list for λ: (NO embeddings !)

Let $\lambda' = w^{-1} \cdot \lambda$, $\lambda_k = (s_k \cdots s_1) \cdot \lambda'$. $\quad (\lambda_{k+1} = s_{k+1} \cdot \lambda_k)$

③ Without loss of generality, we can assume $\mu < \lambda$. $(\mu_k \neq \lambda_k)$

Let $w_k = s_n \cdots s_{k+1}$ $(w_n = 1)$. Then $\mu_k = w_k^{-1} \cdot \mu$ and $\lambda_k = w_k^{-1} \cdot \lambda$

Thus, $\mu_k = w_k^{-1} \cdot \mu = w_k^{-1} s_\alpha \cdot \lambda = s_{w_k^{-1}(\alpha)} \cdot \lambda_k = s_{\beta_k} \cdot \lambda_k$, where $\beta_k := w_k^{-1}(\alpha)$.

Then $\mu_k - \lambda_k = -\langle \lambda_k + \rho, \beta_k^\vee \rangle \beta_k$ is a negative integral multiple of β_k.

since $-\langle \lambda_k + \rho, \beta_k \rangle = -\langle w_k^{-1}(\lambda+\rho), w_k^{-1}(\alpha) \rangle = -\langle \lambda + \rho, \alpha \rangle < 0$ $\quad (\mu - \lambda < 0)$.

We have $\mu_n = \mu < \lambda = \lambda_n$ and $\mu_0 = \mu' > \lambda' = \lambda_0$. Thus, there exists k s.t.

$\mu_{k+1} < \lambda_{k+1}$ and $\mu_k > \lambda_k$, i.e.

$$\begin{array}{ccccccccc} M(\lambda') = M(\lambda_0) & M(\lambda_1) & \cdots & M(\lambda_k) & & M(\lambda_{k+1}) & \cdots & M(\lambda_n) = M(\lambda) \\ \wedge & \cdots & & \wedge & & \vee & \cdots & \vee \\ M(\mu') = M(\mu_0) \supseteq & M(\mu_1) & \supseteq \cdots & M(\mu_k) & \supseteq & M(\mu_{k+1}) \supseteq & \cdots & M(\mu_n) = M(\mu) \end{array}$$

④ Note that $\underbrace{\mu_{k+1} - \lambda_{k+1}}_{<0} = s_{k+1} \cdot (\underbrace{\mu_k - \lambda_k}_{>0}) \Rightarrow \beta_k = -\alpha_{k+1}$, $\beta_{k+1} = \alpha_{k+1}$

By Prop 5.4, $M(\mu_{k+1}) \subseteq M(\lambda_{k+1})$, because $\lambda_{k+1} - s_{k+1} \cdot \lambda_{k+1} = \lambda_{k+1} - \mu_{k+1} > 0$.

Then $M(\mu) \subseteq M(\mu_{n-1}) \cdots \subseteq M(\mu_{k+1}) \subseteq M(\lambda_{k+1})$. Prop 4.5 is naturally needed here

Now consider $M(\lambda_{k+2}) = M(s_{k+2} \cdot \lambda_{k+1})$, by Prop 4.5.

$M(\lambda_{k+1}) \subseteq M(\lambda_{k+2})$ or $M(\mu_{k+2}) = M(s_{k+2} \cdot \mu_{k+1}) \subseteq M(\lambda_{k+2}) \subseteq M(\lambda_{k+1})$,

Thus, we always have $M(\mu_{k+2}) \subseteq M(\lambda_{k+2})$. Therefore, $M(\mu) \subseteq M(\lambda)$.

4.7 Existence of Embeddings: General Case

Fix $\alpha\in\Phi^+$, $n\in\mathbb{Z}^{>0}$.

Denote $X=\{\lambda\in\mathfrak{h}^*: M(\lambda-n\alpha)\subset M(\lambda)\}$, $H=\{\lambda\in\mathfrak{h}^*:\langle\lambda+\rho,\alpha^\vee\rangle=n\}$.

Previously, we have proved that $\Lambda\cap H\underbrace{\subset}_{\text{integral}} X\underbrace{\subset}_{\text{obv.}} H$

Moreover, $\Lambda\cap H$ is Zaraski dense in H. (

Thus, it suffices to show X is Zaraski closed in H.

Idea: Find a $g^\lambda: \mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\to\mathcal{U}(\mathfrak{n}^-)^\ell$ s.t.

Kostant's partition function

$$\lambda\in X\iff \operatorname{rank} g^\lambda<\dim\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\iff \text{all } P(n\alpha)\times P(n\alpha) \text{ minors have det } 0.$$

Thm. $X=H$.

Pf. Only need to show X is a Zaraski close set.

Let $\lambda=\sum c_i\varpi_i\in X$, where ϖ_is are fundamental weights. Our goal is to find a polynomial condition for c_i's.

Take standard bases (h_i,x_i,y_i) for copies S_i of $\mathfrak{sl}_2(\mathbb{C})$ corresponding to simple roots.

For any $u\in\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}$,

$$[x_i,u]=x_iu-ux_i=u_i+u_i'h_i,$$

where u_i and u_i' have degree $-n\alpha+\alpha_i$.

Define $f_i^\lambda:\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\to\mathcal{U}(\mathfrak{n}^-)$ by $f_i^\lambda(u)=u_i+u_i'\lambda(h_i)=u_i+u_i'c_i$.

Note that 1. u_i and u_i' independ on c_i.

2. f_i^λ actually maps $\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}$ into $\mathcal{U}(\mathfrak{n}^-)_{-n\alpha+\alpha_i}$

Combining f_i^λ's gives $g^\lambda:\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\to\bigoplus_{i=1}^{\ell}\mathcal{U}(\mathfrak{n}^-)_{-n\alpha+\alpha_i}$

$$\begin{aligned}
\lambda\in X &\iff \exists v\in M(\lambda)_{\lambda-n\alpha} \text{ s.t. } \mathfrak{n}\cdot v=\mathfrak{n}\,u\cdot v^+=0 \text{ for some } u\in\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\\
&\iff x_1u\cdot v^+=x_2u\cdot v^+=\cdots=x_\ell u\cdot v^+=0 \text{ for some } u\in\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\\
&\iff f_i^\lambda(u)v^+=0,\ \forall i. \text{ for some } u\in\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}\\
&\iff g^\lambda(u)=0, \text{ for some } u\in\mathcal{U}(\mathfrak{n}^-)_{-n\alpha} \text{ (nonzero!)}\\
&\iff \operatorname{rank} g^\lambda<\dim\mathcal{U}(\mathfrak{n}^-)_{-n\alpha}
\end{aligned}$$

Thus, viewing g^λ as a matrix gives a $\sum P(n\alpha-\alpha_i)\times P(n\alpha)$ matrix with entries being linear funtion on c_i's.

Therefore, $\lambda \in X \Leftrightarrow$ all $P(n\alpha)$-th minors (finitely many, say K) have determinate 0.

$$\Leftrightarrow \varphi_1(c) = \varphi_2(c) = \cdots = \varphi_K(c) = 0$$

4.8 Simplicity Criterion: General Case.

Thm. Let $\lambda \in \mathfrak{h}^*$, Then $M(\lambda)$ is simple iff λ is antidominant.

Pf. "$\Rightarrow$" λ antidominant $\Leftrightarrow$ $\forall \alpha \in \Phi^+, \langle \lambda+\rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$

Suppose it fails for some α. Then $s_\alpha \cdot \lambda = \lambda - \langle \lambda+\rho, \alpha^\vee \rangle \alpha < \lambda$.

By Verma Theorem, $M(s_\alpha \cdot \lambda) \subsetneq M(\lambda)$ which conflicts with simplicity of $M(\lambda)$.

"$\Leftarrow$" Almost the same as integral case (substitude W to $W_{[\lambda]}$)

Cor. Let $\lambda \in \mathfrak{h}^*$ anti-dominant. Then for all $w \in W_{[\lambda]}$, the socle of $P(w\cdot\lambda)$ is a direct sum of copies of $L(\lambda)$.

Pf. By Thm 3.10, there is a standard filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_n = P(w\cdot\lambda).$$

with $P_i/P_{i-1} \simeq M(w'\cdot\lambda)$ for some $w' \in W_{[\lambda]}$.

Let L be a simple summand of $\operatorname{soc} P(w\cdot\lambda)$ and i such that $L \subset P_i$ and $L \cap P_{i-1} = 0$.

Thus $L \hookrightarrow M(w'\cdot\lambda) \Rightarrow L \simeq L(\lambda)$.

Ex. Let $\lambda \in \mathfrak{h}^*$, If $P(\lambda) \simeq P(\lambda)^\vee$ is self-dual, i.e. $P(\lambda) \simeq Q(\lambda)$. Prove that λ must be antidominant. What can we say about the converse.

Pf. Suppose μ is the antidominant weight in the linkage class of λ

Then $\operatorname{soc} P(\lambda) = \bigoplus L(\mu)$.

Moreover, $L(\lambda) \hookrightarrow Q(\lambda) = P(\lambda)$.

Thus, $L(\lambda) \hookrightarrow \operatorname{soc} P(\lambda) = \bigoplus L(\mu)$, implies $L(\lambda) = L(\mu)$ by Schur's lemma.

$\Rightarrow \lambda = \mu$.

4.9 Blocks of $\mathcal{O}$

Recall: $\mathcal{O} = \bigoplus_{\chi} \mathcal{O}_\chi$ and for each character χ, we can find a weight λ s.t. $\chi = \chi_\lambda$.
Moreover, by Harish-Chandra's thm, $\chi_\lambda = \chi_\mu$ if and only if $\lambda \in W \cdot \mu$.

Set $B_\lambda := \{ M \in \mathcal{O} : \text{composition factors of } M \in W_{[\lambda]} \cdot \lambda \}$ and

$\mathcal{B}_\lambda$ the full subcat consisting of B_λ.

Thm. 1. $\mathcal{B}_\lambda$ is a block if λ is anti-dominant.

2. $\mathcal{O} = \bigoplus_{\lambda \text{ anti-dominant}} \mathcal{B}_\lambda$

Rmk. It gives a bijection: $\{\text{anti-dominants}\} \longleftrightarrow \{\text{blocks of } \mathcal{O}\}$

pf. 1. It suffices to show $L(w \cdot \lambda)$, $w \in W_{[\lambda]}$ are in the same block.

If $L(\mu)$ is a simple module in $\mathcal{B}_\lambda$, then $\mu \in W_{[\lambda]} \cdot \lambda$.

By antidominance of λ, $L(\lambda) = M(\lambda) \subset M(\mu)$.

Thus, $L(\mu) = M(\mu)/N(\mu)$ and $L(\lambda) \subset N(\mu)$, i.e. $L(\mu)$ and $L(\lambda)$ are in the same block.

2. ① For any simple module $L(\nu)$ in $\mathcal{O}$, let $L(\lambda)$ be the simple socle of $M(\nu)$.

By Simplicity Criterion, λ is anti-dominant. Thus, $L(\nu) \in \mathcal{B}_\lambda$.

② If λ, μ are anti-dominant weights s.t. $B_\lambda = B_\mu$, then $\mu \in W_{[\lambda]} \cdot \lambda$

Thus, $\lambda \leq \mu$. By antidominance of μ, $\mu \leq \lambda$. $\Rightarrow \mu = \lambda$.

Rmk.
- We denote block $\mathcal{B}_\lambda$ associated to an anti-dominant λ by $\mathcal{O}_\lambda$
- If a linkage class decomposes as the following:

$$W \cdot \lambda = W_{[\lambda_1]} \cdot \lambda_1 \sqcup W_{[\lambda_2]} \cdot \lambda_2 \sqcup \cdots \sqcup W_{[\lambda_k]} \lambda_k$$

Then $|W_{[\lambda_i]} \cdot \lambda_i|$ are the same (hence $\mathcal{O}_{\lambda_i}$ has the same number of simple mod)

By Mathieu's paper, $\mathcal{O}_{\lambda_i} \simeq \mathcal{O}_{\lambda_j}$ categorical equivalence.

Ex. If $M = M^{\lambda_1} \oplus \cdots \oplus M^{\lambda_k}$ (block decomp.) has a contra form, then $M^{\lambda_i} \perp M^{\lambda_j}$. (By 3.14(a))

4.10 Example antidominant projectives

Recall: ① $L(\lambda)^\vee = L(\lambda)$. (Thm 3.2(c))

② If $M \in \mathcal{O}$, L fin dim, then $(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$ (Exercise 3.2)

③ If L fin dim, then $M(\lambda) \otimes L$ has a std filtration with factors being $M(\lambda+\mu)$, where μ runs over weight of L, and with multiplicity $\dim L_\lambda$.
(Thm 3.6)

④ Suppose $\lambda \in \mathfrak{h}^*$ dominant, then $M(\lambda)$ is projective.

⑤ If P projective, L fin dim, then $P \otimes L$ is projective in $\mathcal{O}$. (Prop 3.8)

Thm. Let $\lambda+\rho \in \Lambda^+$, so $w_0 \cdot \lambda$ is anti-dominant and integral. Then

1. the std fil of $P(w_0 \cdot \lambda)$ involves all of distinct $M(w \cdot \lambda)$ exactly once.
2. $[M(w \cdot \lambda) : L(w_0 \cdot \lambda)] = 1$ for all $w \in W$
3. $P(w_0 \cdot \lambda)$ is self-dual.

Pf. 1. Consider $T = M(-\rho) \otimes L(\lambda+\rho) = L(-\rho) \otimes L(\lambda+\rho)$

By ④ & ⑤, T is projective. Then the direct summand T^{χ_λ} is also projective.

By ③, T has a std filtration with factors being $M(\mu-\rho)$, where μ ranges over weight of $L(\lambda+\rho)$, and with multiplicity 1. Thus, T^{χ_λ} also has a std fil. with factors $M(\mu-\rho)$, where $\mu-\rho \in W \cdot \lambda$ and is a weight of $L(\lambda+\rho)$.

Thus, $[T^{\chi_\lambda}] = \sum_{w \in W} [M(w \cdot \lambda)]$. (multiplicity 1)

Note that T^{χ_λ} has a quotient $M(w_0 \cdot \lambda)$ (by Rmk 3.6) and $M(w_0 \cdot \lambda)$ is simple since $w_0 \cdot \lambda$ anti-dominant. Thus the projective cover $P(w_0 \cdot \lambda)$ must be a summand of T^{χ_λ}. (Essential surjectivity)

Therefore, $P(w_0 \cdot \lambda)$ has a std fil involving various modules $M(w \cdot \lambda)$ with multiplicity 1.

2. By BGG Reciprocity,

$[M(w \cdot \lambda) : L(w_0 \cdot \lambda)] = (P(w_0 \cdot \lambda) : M(w \cdot \lambda)) \leq (T^{\chi_\lambda} : M(w \cdot \lambda)) = 1$

On the other hand, $M(w \cdot \lambda)$ has a unique simple socle $L(w_0 \cdot \lambda)$

Thus, $[M(w\cdot\lambda) : L(w_0\cdot\lambda)] = 1$.

3. By 2, $(P(w_0\cdot\lambda) : M(w\cdot\lambda)) = (T^{\chi_\lambda} : M(w\cdot\lambda)) = 1$. Thus, $P(w_0\cdot\lambda) = T^{\chi_\lambda}$.

Note that $T^\vee \underset{②}{\cong} M(-\rho)^\vee \otimes L(\lambda+\rho)^\vee \underset{①}{\cong} T$. Then T self-dual.

Then T is projective and injective, so is its direct summand T^{χ_λ}

Since $L(w_0\cdot\lambda) \hookrightarrow M(\lambda) \hookrightarrow T^{\chi_\lambda}$, then $Q(w_0\cdot\lambda) \subset T^{\chi_\lambda}$

Then by injectivity of $Q(w_0\cdot\lambda)$, there is a splitting map $T^{\chi_\lambda} \to Q(w_0\cdot\lambda)$

By indecomposibility of T^{χ_λ}, $Q(w_0\cdot\lambda) = T^{\chi_\lambda}$. $\left(\begin{tikzcd} Q \arrow[r, hook] \arrow[d, "id"'] & T \arrow[dl] \\ Q & \end{tikzcd}\right)$

Thus, $P(w_0\cdot\lambda)^\vee = Q(w_0\cdot\lambda) = P(w_0\cdot\lambda)$.

Ex. Under the hypothesis of the theorem, what is $\dim \operatorname{End} P(w_0\cdot\lambda)$

pf. $\dim \operatorname{End} P(w_0\cdot\lambda) = |W|$

By Thm 3.9, $\dim \operatorname{End} P(\lambda) = [P(\lambda) : L(\lambda)]$ for any λ.

Thus, $\dim \operatorname{End} P(w_0\cdot\lambda) = [P(w_0\cdot\lambda) : L(w_0\cdot\lambda)] = |W|$

4.11 Application to $\mathfrak{sl}_3(\mathbb{C})$

Recall: Weyl group of $\mathfrak{sl}_3(\mathbb{C})$: $\{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha = w_0\}$

where α, β are simple roots of $\mathfrak{sl}_3(\mathbb{C})$.

Let λ be a regular, integral and antidominant weight.

Then $M(\lambda)$ is simple and every $M(w\cdot\lambda)$ contains unique simple submodule $M(\lambda)$

By Thm 4.10, $[M(w\cdot\lambda) : L(\lambda)] = 1$.

- $M(\lambda)$ simple
- $M(s_\alpha\cdot\lambda)$ has composition factors $L(\lambda)$ & $L(s_\alpha\cdot\lambda)$ (multiplicity 1)
- $M(s_\beta\cdot\lambda)$ - - - - - & $L(s_\beta\cdot\lambda)$ - - -

Ex. In $\mathfrak{sl}_3(\mathbb{C})$, what can be said at this pt about singular integral highest weight

Let $\{\alpha, \beta\}$ be simple rts and $\varpi_\alpha, \varpi_\beta$ be foundamental weights

Then $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$. It is clearly that $\varpi_\alpha = \frac{2}{3}\alpha + \frac{1}{3}\beta$, $\varpi_\beta = \frac{1}{3}\alpha + \frac{2}{3}\beta$

(Conversely, $\alpha = 2\varpi_\alpha - \varpi_\beta$, $\beta = 2\varpi_\beta - \varpi_\alpha$)

Let $\lambda = a\varpi_\alpha + b\varpi_\beta$, $a, b \in \mathbb{Z}$.

$$s_\alpha \cdot \lambda = s_\alpha(a\varpi_\alpha + b\varpi_\beta + \rho) - \rho = a\varpi_\alpha + b\varpi_\beta - \langle a\varpi_\alpha + b\varpi_\beta + \rho, \alpha^\vee \rangle \alpha$$

$$= a\varpi_\alpha + b\varpi_\beta - (a+1)\alpha = \lambda - (a+1)\alpha$$

$$s_\beta \cdot \lambda = \lambda - (b+1)\beta$$

$$s_\beta s_\alpha \cdot \lambda = s_\beta \cdot (\lambda - (a+1)\alpha) = s_\beta(\lambda - (a+1)\alpha + \rho) - \rho = s_\beta \cdot \lambda - (a+1)s_\beta(\alpha)$$

$$= \lambda - (b+1)\beta - (a+1)(\alpha+\beta) = \lambda - (a+1)\alpha - (a+b+2)\beta$$

$$s_\alpha s_\beta \cdot \lambda = \lambda - (a+b+2)\alpha - (b+1)\beta$$

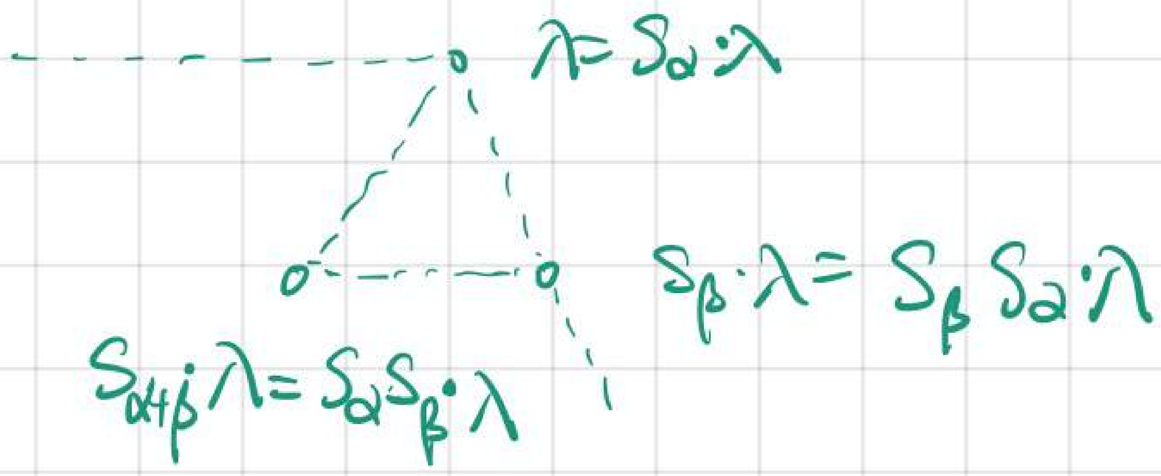
$$s_\alpha s_\beta s_\alpha \cdot \lambda = s_\alpha \cdot (\lambda - (a+1)\alpha - (a+b+2)\beta) = \lambda - (a+b+2)(\alpha+\beta)$$

Note that $w_0 = s_{\alpha+\beta} = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$

- If $s_\alpha \cdot \lambda = s_\beta \cdot \lambda$, then $a = b = -1$, $\lambda = -\rho$. and $|W \cdot \lambda| = 1$. Not interesting !!

Let's consider $|W \cdot \lambda| \neq 1$.

- If $\lambda = s_\alpha \cdot \lambda$, i.e. $a = -1$, $b \neq -1$. (λ lies in α-hyperplane) Figure ①



linkage class : $\{\lambda, \underset{\lambda-(b+1)\beta}{\underset{\parallel}{s_\beta \cdot \lambda}}, \underset{\lambda-(b+1)(\alpha+\beta)}{\underset{\parallel}{s_\alpha s_\beta \cdot \lambda}}\}$

① $b+1 \in \mathbb{Z}^{>0}$: $M(s_\alpha s_\beta \cdot \lambda) \subset M(s_\beta \cdot \lambda) \subset M(\lambda)$

Note that $s_\alpha s_\beta \cdot \lambda$ is antidominant and integral. Then by Thm 4.10,

$$[M(s_\alpha \cdot \lambda) : M(s_\alpha s_\beta \cdot \lambda)] = [M(\lambda) : M(s_\alpha s_\beta \cdot \lambda)] = 1.$$

Then $0 \to L(s_\alpha s_\beta \cdot \lambda) \to M(s_\alpha \cdot \lambda) \to L(s_\alpha \cdot \lambda) \to 0$ SES

Let $0 \to M(s_\beta \cdot \lambda) \to M(\lambda) \to L \to 0$.

Then L has composition factors $L(\lambda), L(s_\beta \cdot \lambda), L(s_\alpha s_\beta \cdot \lambda)$ with mult m_1, m_2, m_3 resp.

- $m_3 = 0$ ($M(s_\beta \cdot \lambda)$ has $L(s_\alpha s_\beta \cdot \lambda)$ as mult 1, but $[M(\lambda) : L(s_\alpha s_\beta \cdot \lambda)] = 1$).
- $m_1 = 1$. $m_2 = 0$. ($\dim M(\lambda)_\lambda = \dim M(\lambda)_{s_\beta \cdot \lambda} = 1$)

Thus $L = L(\lambda)$, i.e. $0 \to L(s_\alpha s_\beta \cdot \lambda) \to M(s_\alpha \cdot \lambda) \to M(\lambda) \to 0$ is a composition series of $M(\lambda)$.

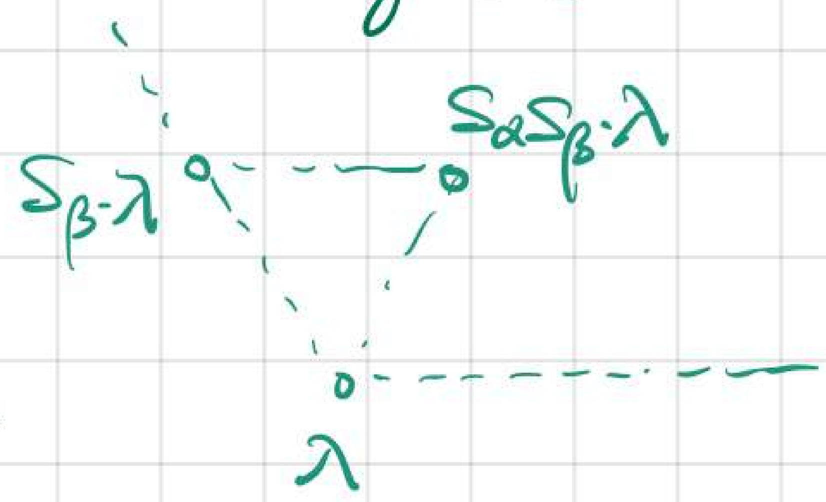
② $b+1 \in \mathbb{Z}^{<0}$: $M(\lambda) \subset M(s_\beta \cdot \lambda) \subset M(s_\alpha s_\beta \cdot \lambda)$

Note that λ is antidominant and integral. By Thm 4.10,

$[M(s_\alpha \cdot \lambda) : M(\lambda)] = [M(s_\alpha s_\beta \cdot \lambda) : M(\lambda)] = 1$

Then $0 \to L(\lambda) \to M(s_\beta \cdot \lambda) \to L(s_\beta \cdot \lambda) \to 0$ is SES.

Figure ②



Similar analysis as previous case ① gives

$0 \to M(s_\beta \cdot \lambda) \to M(s_\alpha s_\beta \cdot \lambda) \to L(s_\alpha s_\beta \cdot \lambda) \to 0$ is SES

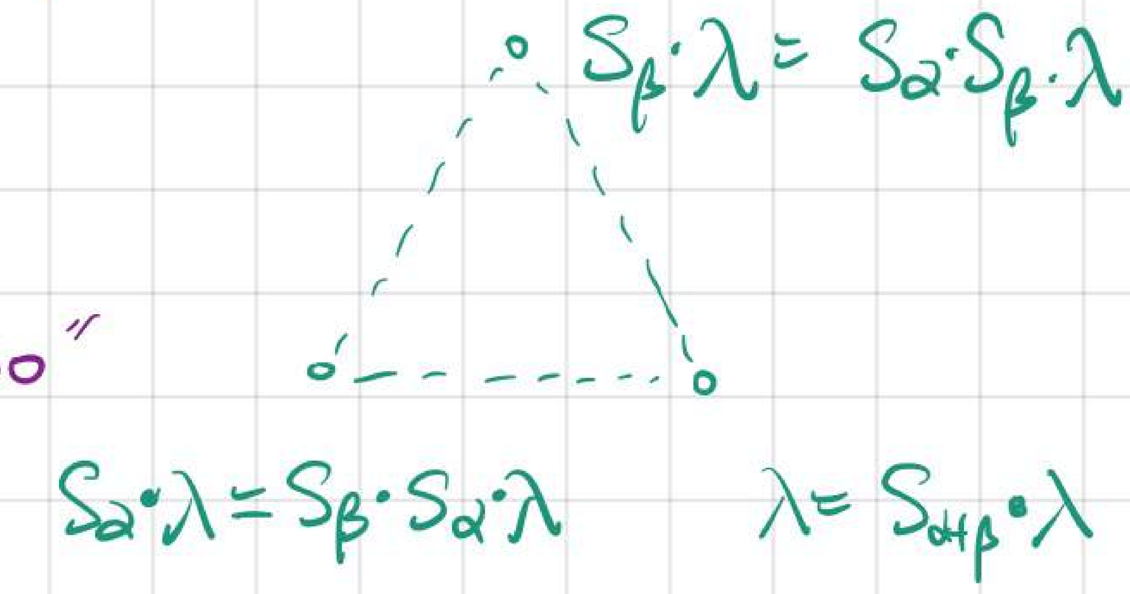
and $0 \to M(\lambda) \to M(s_\beta \cdot \lambda) \to M(s_\alpha s_\beta \cdot \lambda) \to 0$ is a composition serise of $M(s_\alpha s_\beta \cdot \lambda)$

- If $\lambda = s_\beta \cdot \lambda$, i.e. $a \neq -1$, $b = -1$. (λ lies in β-hyperplane)

  linkage class: $\{\lambda, s_\alpha \cdot \lambda, s_\beta s_\alpha \cdot \lambda\}$

Actually we do not need to consider it again, since λ is just "$s_\alpha s_\beta \cdot \lambda$" in the previous case "$\lambda = s_\alpha \cdot \lambda$".

Figure "$a+1 \in \mathbb{Z}^{>0}$"



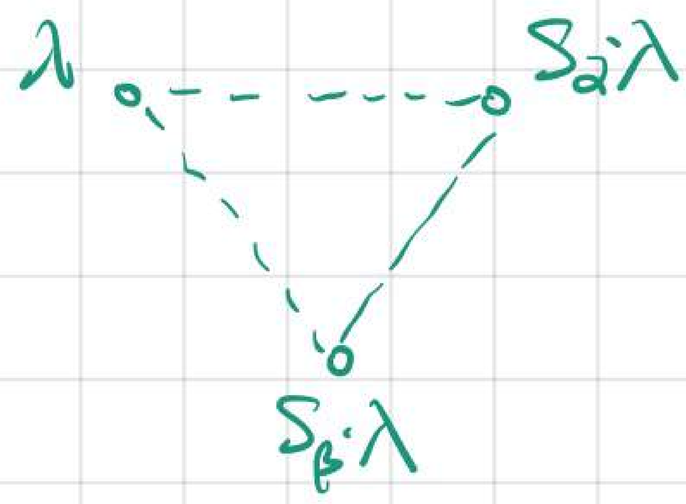
- If $\lambda = s_\alpha s_\beta s_\alpha \cdot \lambda$, then $a+b = -2$. and $(a,b) \neq (-1,-1)$.

  linkage class: $\{\lambda, s_\alpha \cdot \lambda, s_\beta \cdot \lambda\}$

For discussion in detail, we need to consider "$a+1 \in \mathbb{Z}^{>0}$" and "$a+1 \in \mathbb{Z}^{<0}$" separately.

Actually, we do not have to do so, because λ here is just "$s_\beta \cdot \lambda$" in previous case "$\lambda = s_\alpha \cdot \lambda$".

Figure "$a+1 \in \mathbb{Z}^{<0}$"



4.12. Shapovalov Elements

Background: • In Verma thm, we know that there exists an embedding $M(s_\gamma\cdot\lambda)\hookrightarrow M(\lambda)$ iff $s_\gamma\cdot\lambda<\lambda$, $\gamma\in\Phi^+$. By Lemma 4.2 ($\dim \mathrm{Hom}(M(\mu), M(\lambda))\leq 1$) we know that this embedding is unique up to scalar.

• In proposition 1.4, we can construct the embedding explicitly when $\gamma\in\Delta^+$.

So the natural question is as following:

Is is possible to construct the embedding explicitly for any $\gamma\in\Phi^+$?

Main difficulties: $\dim U(\mathfrak{n}^-)_{-\lambda+s_\gamma\cdot\lambda} = P(-\lambda+s_\gamma\cdot\lambda)$ which is usually too large.

Let's think about it by an example:

$\mathfrak{g}=\mathfrak{sl}_3$, $\Phi^+=\{\alpha,\beta,\gamma=\alpha+\beta\}$, standard basis $=\{x_\alpha, x_\beta, x_\gamma, h_\alpha, h_\beta, y_\alpha, y_\beta, y_\gamma\}$

• Set $\nu=\gamma\in P$ (because $\varpi_\alpha=\frac{2}{3}\alpha+\frac{1}{3}\beta$, $\varpi_\beta=\frac{1}{3}\alpha+\frac{2}{3}\beta$)

$U(\mathfrak{n}^-)_{-\nu}=\mathrm{span}\{y_\alpha y_\beta, y_\gamma\}$

• Let $\lambda=a\varpi_\alpha+b\varpi_\beta$ with $\langle\lambda+\rho,\gamma^\vee\rangle=1$, i.e. $a+b=-1$. Then $\lambda-s_\alpha\cdot\lambda=\gamma$.

• Set undetermined coefficients $r,s\in\mathbb{C}$, s.t. nonzero $u=r\,y_\alpha y_\beta+s\,y_\gamma\in U(\mathfrak{n}^-)_{-\nu}$ satisfies $u\cdot v^+$ is a maximal vector, i.e. $x_\alpha u\cdot v^+=x_\beta\cdot u\cdot v^+=0$ ---- $(*)$

• Note $x_\alpha y_\alpha y_\beta = y_\alpha x_\alpha y_\beta + h_\alpha y_\beta = y_\alpha y_\beta x_\alpha + y_\beta h_\alpha + y_\beta$

$x_\alpha y_\gamma = y_\gamma x_\alpha - y_\beta$

$x_\beta y_\alpha y_\beta = y_\alpha y_\beta x_\beta + y_\alpha h_\beta$

$x_\beta y_\gamma = y_\gamma\cdot x_\beta + y_\alpha$

Then $(*)\iff r(a y_\beta v^+ + y_\beta v^+) - s y_\beta v^+=0$ & $r b y_\alpha v^+ + s y_\alpha v^+=0$

$\iff r(a+1)-s=0$ & $rb+s=0$ $\quad (a+b=-1, (r,s)\neq(0,0))$

$\Rightarrow r\neq 0$ & $rb+s=0$,

Thus, we can get $(r,s)=(1,-b)$ by a nonzero scalar which implies

$u:=y_\alpha y_\beta - y_\gamma h_\beta\in U(\mathfrak{b}^-)_{-\gamma}$ and $u v^+$ is always a maximal vector in $M(\lambda)$ as long as $\langle\lambda+\rho,\gamma^\vee\rangle=1$.

From this example, we obtain experience:

- $\exists\, u \in \mathcal{U}(\mathfrak{n}^-)_{-\gamma}$ which produces a maximal vector $u v^+$ but related to λ, and u is unique up to scalar.
- $\exists\, u \in \mathcal{U}(\mathfrak{b}^-)_{-\gamma}$ which also produces a maximal vector $u v^+$ but indep. on λ, as long as λ lies on a hyperplane ($\langle \lambda+\rho, \gamma^\vee \rangle = \overset{\rightsquigarrow 1}{k} \in \mathbb{Z}^{>0}$), but this u is no longer unique, it's actually unique modulo the left ideal generated by $h_\gamma + \rho(h_\gamma) - \overset{\rightsquigarrow 1}{k}$.

Thm (Shapovalov) Fix $\gamma \in \Phi^+$ and an integer $r > 0$. There exists an element $\Theta_{\gamma,r} \in \mathcal{U}(\mathfrak{b}^-)_{-r\gamma}$ having the following properties:

a) For each root $\beta > 0$, the commutator $[x_\beta, \Theta_{\gamma,r}]$ lies in the left ideal

$I_{\gamma,r} := \mathcal{U}(\mathfrak{g})(h_\gamma + \rho(h_\gamma) - r) + \mathcal{U}(\mathfrak{g})\mathfrak{n}$

b) If $\gamma = \sum_{i=1}^{l} a_i \alpha_i$, we can write $\Theta_{\gamma,r} = \prod_{i=1}^{l} y_i^{r a_i} + \sum_j p_j q_j$, with $p_j \in \mathcal{U}(\mathfrak{n}^-)_{-r\gamma}$, $q_j \in \mathcal{U}(\mathfrak{h})$ and $\deg p_j < r \sum_i a_i$.

Moreover, $\Theta_{\gamma,r}$ is unique (up to scalar) modulo the left ideal $J_{\gamma,r} := \mathcal{U}(\mathfrak{b}^-)(h_\gamma + \rho(h_\gamma) - r)$

Rmk. In (b), note that the highest term doesn't depend on the ordering of simple rts.

4.13 Proof of Shapovalov's Theorem

This section is based on Shapovalov's paper "On a bilinear form on the ..."

Step 1. Induction on $\mathrm{ht}\,\gamma$

The proof will use induction on $\mathrm{ht}\,\gamma$:

If $\mathrm{ht}\,\gamma = 1$, i.e. γ is a simple root, then Proposition 1.4 gives $\Theta_{\gamma,r} = y_\gamma^r$.

Assume $\gamma \notin \Delta$, there exists $\alpha \in \Delta$ s.t. $p := \langle \gamma, \alpha^\vee \rangle > 0$. Now fix γ & α.

Denote $\gamma' := s_\alpha(\gamma) = \gamma - p\alpha < \gamma$. By induction hypo, $\exists\, \Theta_{\gamma',r}$ with desired condition.

Step 2. Dense subset.

Since we are only interested in those λ for which $M(\lambda - r\gamma) \hookrightarrow M(\lambda)$, we consider

$$H_{\gamma,r} := \{\lambda \in \mathfrak{h}^* : \langle \lambda+\rho, \gamma^\vee \rangle = r\}$$

We know that $\Lambda \cap H_{\gamma,r}$ is Zariski-dense in $H_{\gamma,r}$ and so is

$$\Theta = \Lambda \cap H_{\gamma,r} \cap H_\alpha = \Lambda \cap H_{\gamma,r} \cap \{\lambda \in \mathfrak{h}^* : \langle \lambda+\rho, \alpha^\vee \rangle < 0\}.$$

Denoted by $-q$

i.e. $\Theta = \Lambda \cap \{\lambda \in \mathfrak{h}^* : s_\gamma \cdot \lambda = \lambda - r\gamma < \lambda \ , \ s_\alpha \cdot \lambda = \lambda + q\alpha < \lambda\}$

Step 3. Sufficient condition

For $\lambda \in \Theta$, associate each $\theta \in \mathcal{U}(\mathfrak{b}^-)$ an element $\theta(\lambda)$, which is the unique element in $(\theta + I_\lambda) \cap \mathcal{U}(\mathfrak{n}^-)$, where $I_\lambda := \sum_{h \in \mathfrak{h}} \mathcal{U}(\mathfrak{g})(h - \lambda(h)) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$, a left ideal which annihilates maximal vector v^+ in $M(\lambda)$.

Lemma 4.13

To prove the thm, it suffices to construct a family of $\theta_{\gamma,r}(\lambda) \in \mathcal{U}(\mathfrak{n}^-)_{-r\gamma}$ for any $\lambda \in \Theta$, satisfying the following conditions:

a') $[x_\beta, \theta_{\gamma,r}(\lambda)] \in I_\lambda$ for all $\beta \in \Phi^+$

b') the highest term of $\theta_{\gamma,r}(\lambda)$ (relative to natural filtration of $\mathcal{U}(\mathfrak{n}^-)$) is $\prod_i y_i^{r a_i}$ (independent of the ordering and λ)

c') the coefficients of $\theta_{\gamma,r}(\lambda)$ in PBW basis depend polynomially on λ (hence in any basis)

* c') means: relative to the fixed basis of $\mathcal{U}(\mathfrak{n}^-)$, there are $p_j \in \mathcal{U}(\mathfrak{n}^-)_{-r\gamma}$ and $q_j \in \mathcal{U}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}]$ (indep. on λ) s.t. $\theta_{\gamma,r}(\lambda) = \sum_j p_j \cdot q_j(\lambda)$.
(Here q_j is viewed as polynomial function on $\mathfrak{h}^*$, i.e. $q_j : \mathfrak{h}^* \xrightarrow[\text{poly}]{} \mathbb{C}$.)

Now we show how this family of $\theta_{\gamma,r}(\lambda)$ implies the thm:

Consider $\theta_{\gamma,r} := \sum p_j \cdot q_j \in \mathcal{U}(\mathfrak{b}^-)_{-r\gamma}$

a): $\forall \beta \in \Phi^+$, $[x_\beta, \theta_{\gamma,r}(\lambda)] = -\theta_{\gamma,r} x_\beta + \sum_j x_\beta p_j q_j(\lambda) \in I_\lambda \quad \forall \lambda \in \Theta$

$\Rightarrow [x_\beta, \theta_{\gamma,r}] \in I_\lambda \ , \ \forall \lambda \in \Theta$ — $\theta_{\gamma,r}(\lambda)$ acts on $M(\lambda)$ the same as $\theta_{\gamma,r}$

$\Rightarrow [x_\beta, \theta_{\gamma,r}] \in I_\lambda \ , \ \forall \lambda \in H_{\gamma,r}$ — Zariski-dense

$\Rightarrow [x_\beta, \theta_{\gamma,r}] \in \bigcap_{\lambda \in H_{\gamma,r}} I_\lambda.$

So now it suffices to prove $\bigcap_{\lambda\in H_{\gamma,r}} I_\lambda = I_{\gamma,\lambda}$.

"$\supseteq$" is easy because $I_{\gamma,\lambda} \subseteq I_\lambda$

Conversely, the proof involves Hilbert Nullstellensatz thm.

Note that

$$I_\lambda = \sum_{h\in\mathfrak{h}} \mathcal{U}(\mathfrak{b}^-)(h-\lambda(h)) + \sum \mathcal{U}(\mathfrak{g})\,\mathfrak{n}^+$$

$$I_{\gamma,r} = \mathcal{U}(\mathfrak{b}^-)(h_\gamma + \rho(h_\gamma) - r) + \sum \mathcal{U}(\mathfrak{g})\,\mathfrak{n}^+$$

So we only have to consider the terms in $\mathcal{U}(\mathfrak{b}^-)$.

Take $X = \sum p_j q_j \in \bigcap_{\lambda\in H_{\gamma,r}} I_\lambda$, where p_j are a basis of $\mathcal{U}(\mathfrak{n}^-)$ & $q_j \in \mathcal{U}(\mathfrak{h})$

Veiw q_j as polynomials on $\mathfrak{h}^*$. Then q_j vanishes $H_{\gamma,r} = V(h_\gamma + \rho(h_\gamma) - r)$

Since $H_{\gamma,r}$ is an irred affine variety, $q_j \in \mathcal{U}(\mathfrak{h})(h_\gamma + \rho(h_\gamma) - r)$ by Hilbert thm.

Thus $\bigcap_{\lambda\in H_{\gamma,r}} I_\lambda \subset I_{\gamma,r}$ $\qquad$ $q_j \in I(V(I)) = \sqrt{I} = I$

b): Obv.

uniqueness: $\Theta_{\gamma,r}$ satisfies $[x_\beta, \Theta_{\gamma,r}]\cdot v_\lambda^+ = 0$ for any $\lambda \in H_{\gamma,r}$.

Then $\Theta_{\gamma,r}\cdot v_\lambda^+ \in M(\lambda)_{\lambda - r\gamma}$ is a maximal vector.

Thus $\Theta_{\gamma,r}$ is unique modulo left ideal $J_{\gamma,r}$

Step 4. Proof of the lemma.

Now we recall some notations:

- $\gamma \in \Phi^+ \setminus \Delta$, $\alpha \in \Delta$ fixed, $\gamma = \sum_{i=1}^{l} a_i\alpha_i$, $p = \langle \gamma, \alpha^\vee\rangle > 0$,
- $\gamma' := s_\alpha(\gamma) = \gamma - p\alpha$, then $\gamma' = \sum_{i=1}^{l} b_i\alpha_i$ with $b_i = a_i$ except when $\alpha = \alpha_i$, $b_i = a_i - p$.
- $\lambda \in \Theta$, then $-q = \langle \lambda+\rho, \alpha^\vee\rangle < 0$

Thus $0 < \gamma' < \gamma$, by induced hypo, $\exists\ \Theta_{\gamma',r} \in \mathcal{U}(\mathfrak{b}^-)_{-r\gamma'}$ satisfying the theorem.

in particular, all specialization $\Theta_{\gamma',r}(\mu)$ (cf. the begining of Step 3) are defined for $\mu \in H_{\gamma',r}$ and the highest term of $\Theta_{\gamma',r}$ is $\prod_i y_i^{rb_i}$.

Note that $\mu := s_\alpha\cdot\lambda = \lambda + q\alpha > \lambda$ and $\mu \in H_{\gamma',r}$. Then we have an increasing seq. (of dot action)

$$\lambda - r\gamma \xrightarrow{s_\gamma} \lambda \xrightarrow{s_\alpha} \mu$$

$s_{\gamma'}\cdot\mu = \mu - \langle \mu+\rho, \gamma'^\vee\rangle\gamma'$
$= \mu - \langle s_\alpha(\mu+\rho), \gamma^\vee\rangle\gamma' = \mu - r\gamma'$

On the other hand, we have another increasing seq (of dot action)

$$\lambda - r\gamma \xrightarrow{s_\alpha} \mu - r\gamma' \xrightarrow{s_{\gamma'}} \mu$$

$$\begin{aligned} s_\alpha \cdot (\lambda - r\gamma) &= \lambda - r\gamma - \langle \lambda - r\gamma + \rho, \alpha^\vee \rangle \alpha \\ &= \lambda - r\gamma + q\alpha + r\langle \gamma, \alpha^\vee \rangle \alpha \\ &= \mu - r\gamma + rp\alpha \\ &= \mu - r(\gamma - p\alpha) = \mu - r\gamma' \end{aligned}$$

Thus, we have resulting inclusions of Vermas:

$$\begin{tikzcd} & M(\lambda) \arrow[dr, "y_\alpha^q"] & \\ M(\lambda - r\gamma) \arrow[ur, "\theta_{\gamma,r}(\lambda)"] \arrow[dr, "y_\alpha^n"'] & & M(\mu) \\ & M(\mu - r\gamma') \arrow[ur, "\theta_{\gamma',r}(\mu)"'] & \end{tikzcd}$$

Then the embedding $M(\lambda - r\gamma) \longrightarrow M(\lambda)$ is induced by a unique (up to scalar) element $\theta_{\gamma,r}(\lambda) \in U(n^-)_{-r\gamma}$ s.t.

$$\theta_{\gamma,r}(\lambda) \cdot y_\alpha^q = y_\alpha^n \cdot \theta_{\gamma',r}(\mu)$$

It is plain to check $\theta_{\gamma,r}(\lambda)$ satisfies (a'), (b'). For (c') we need to use the formula above:

Use an alternative PBW basis for $U(n^-)$ in which the power of y_α at the right end of each monomial.

Since $U(n^-)$ has no zero divisors, we can cancel y_α^q from both side to get an explicit expression of $\theta_{\gamma,r}(\lambda)$ (This process is indep. on λ).

Then by property (c'), the coef of $\theta_{\gamma',r}(\mu)$, in tandem with the fact that λ depends linearly on μ, insures that all coef of $\theta_{\gamma,r}(\lambda)$ depend poly. on λ.

Chapter 5. Highest Weight Modules II

5.1. BGG Theorem

Def $\forall \lambda, \mu \in \mathfrak{h}^*$, we write $\mu \hat{\uparrow} \lambda$ if $\mu = \lambda$ or $\exists \alpha \in \Phi^+$ s.t. $\mu = s_\alpha \cdot \lambda < \lambda$

μ is strongly linked to λ: if $\mu = \lambda$ or there exist $\alpha_1, \cdots, \alpha_r \in \Phi^+$ s.t.

$$\mu = (s_1 \cdots s_r) \cdot \lambda \hat{\uparrow} (s_2 \cdots s_r) \cdot \lambda \hat{\uparrow} \cdots \hat{\uparrow} s_{\alpha_r} \cdot \lambda \hat{\uparrow} \lambda .$$

and we write $\mu \uparrow \lambda$.

Now we introduce a theorem and prove it in section 5.5 after developing some deep technique (Jantzen filtration and contravariant forms):

Thm. Let $\lambda, \mu \in \mathfrak{h}^*$.

a) (Verma) If μ is strongly linked to λ, then $M(\mu) \hookrightarrow M(\lambda)$. In particular, $[M(\lambda) : L(\mu)] \neq 0$.

b) (BGG) If $[M(\lambda) : L(\mu)] \neq 0$, then μ is strongly linked to λ.

Note that part (a) is just an iteration of Thm 4.6. But part (b) is hard.

Example: An application of Verma thm: if λ is dominant, not necessarily integral. for all $w \in W_{[\lambda]}$, there is an imbedding $M(w \cdot \lambda) \hookrightarrow M(\lambda)$.

Pf. Let $\Delta_{[\lambda]} = \{\alpha_1, \cdots, \alpha_\ell\}$, Then w has a reduced expression $w = s_1 \cdots s_r$, where $s_i = s_{\alpha_i}$, $\alpha_i \in \Delta_{[\lambda]}$. (NOT necessarily in Δ)

Since Prop 1.4 doesn't remain valid here, we should replace it by Verma thm.

By Verma thm, if $\alpha \in \Delta_{[\lambda]} \in \Phi^+$ and $s_\alpha \cdot \lambda < \lambda$, then $M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$.

Moreover, by a similar proof in Thm 4.3, we have an increasing weight chain:

$$w \cdot \lambda = s_1 \cdots s_r \cdot \lambda < s_2 \cdots s_r \cdot \lambda < \cdots < s_r \cdot \lambda < \lambda$$

Then

$$M(w \cdot \lambda) \hookrightarrow M(s_2 \cdots s_r \cdot \lambda) \hookrightarrow \cdots \hookrightarrow M(s_r \cdot \lambda) \hookrightarrow M(\lambda)$$

Rmk. Taken together, the two parts of the thm imply: Let $\lambda, \mu \in \mathfrak{h}^*$

$$[M(\lambda) : L(\mu)] \neq 0 \iff M(\mu) \hookrightarrow M(\lambda) \quad ---- (*)$$

Exercise. Let $\lambda \in \mathfrak{h}^*$. If all $[M(\lambda):L(\mu)]=1$, use $(*)$ to prove that the maximal submod $N(\lambda)$ of $M(\lambda)$ is the sum of the corresponding embedding Verma modules $M(\mu)$. (Examples show however the converse is false.)

Pf. Since $[M(\lambda):L(\mu)]\neq 0$, then $M(\mu)\hookrightarrow M(\lambda)$ $\forall \mu$.

Then $N(\lambda)\supseteq \sum_{\mu\neq\lambda} M(\mu)$.

Moreover, $1\leq [N(\lambda):L(\mu)]\leq [M(\lambda):L(\mu)]=1$ forces $[N(\lambda):L(\mu)]=1$

Since $N(\lambda)$ has the same composition factors with $\sum_{\mu\neq\lambda} M(\mu)$, $N(\lambda)=\sum_{\mu\neq\lambda} M(\mu)$

(Nonexamples: for sl_3, dominant integral λ ?)

5.2 Bruhat Ordering

Notations: $S=\{$ simple reflections $s_\alpha : \alpha\in\Delta\}$

$T=\{$ reflections $s_\alpha : \alpha\in\Phi^+\}$

Def. (Bruhat ordering of Weyl group) For $w, w'\in W$, $t\in T$, we write $w'\xrightarrow{t} w$ or just $w'\to w$, if $w=t\cdot w'$ and $\ell(w')<\ell(w)$.

Extend this relation to a partial order: define $w'<w$ if

$$w'=w_0\to w_1\to\cdots\to w_r=w$$

for some $w_1,\cdots,w_{r-1}$.

Lemma Let λ is antidominant, regular and integral, $\alpha\in\Phi^+$, $w\in W$, then

$$(s_\alpha w)\cdot\lambda<w\cdot\lambda \iff s_\alpha w<w \text{ in Bruhat order.}$$

Pf. If $s_\alpha\cdot(w\cdot\lambda)<w\cdot\lambda$, it is equivalent to $\langle w\cdot\lambda+\rho,\alpha^\vee\rangle\in\mathbb{Z}^{>0}$, i.e.

$$\langle\lambda+\rho,(w^{-1}\alpha)^\vee\rangle=\langle w(\lambda+\rho),\alpha^\vee\rangle\in\mathbb{Z}^{>0}$$

Since λ is antidominant, $w^{-1}\alpha<0$, or $(w^{-1}s_\alpha)\alpha>0$.

By Prop 0.3(4), $(w^{-1}s_\alpha)\alpha>0\iff \ell(s_\alpha(w^{-1}s_\alpha)^{-1})=\ell(w)>\ell(w^{-1}s_\alpha)=\ell(s_\alpha w)$

This means $s_\alpha w\xrightarrow{s_\alpha} w$.

Scrutiny of the steps shows that they are reversible.

Iterating this lemma gives the following proposition.

Prop Let λ antidominant, regular and integral. For all $w', w \in W$,

$$w' \cdot \lambda \uparrow w \cdot \lambda \iff w' < w$$

Then combine with the Thm in last section. We have

Cor. Let $\lambda \in \Lambda$ be antidominant and regular, and let $w, w' \in W$. Then

$$[M(w \cdot \lambda) : L(w' \cdot \lambda)] \neq 0 \iff w' \leq w'$$

5.3 Jantzen Filtration

Jantzen filtration plays a vital role in the proof of BGG thm. We plan to show the consequences, including proof of BGG thm at first and then complete the proof of Jantzen thm.

Thm (Jantzen). Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda)$ has a filtration by submodules:

$$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots$$

with $M(\lambda)^i = 0$ for large enough i, satisfying:

a) Each nonzero quotent has a nondegenerate contravariant form.
b) $M^1(\lambda) = N(\lambda)$, the unique maximal submodule of $M(\lambda)$
c) The formal characters satisfy

$$\sum_{i>0} \operatorname{ch}(M(\lambda)^i) = \sum_{\alpha>0,\, s_\alpha \cdot \lambda < \lambda} \operatorname{ch} M(s_\alpha \cdot \lambda)$$

Rmk. The filtration is called Jantzen filtration, while the formula in part (c) is called Jantzen sum formula.

Notations.
- $M(\lambda)_i := M(\lambda)^i / M(\lambda)^{i+1}$ the ith filtration layer.
- $\Phi^+_\lambda := \{\alpha \in \Phi^+ \cap \Phi_{[\lambda]} : s_\alpha \cdot \lambda < \lambda\}$, which is precisely what the summation in RHS of part (c) runs over.

Rmk. Suppose we have: For antidominant λ, $[M(w\cdot\lambda):L(\lambda)]=1$ for all $w\in W_{[\lambda]}$.
(We have already proved in the case of $\lambda\in\Lambda$ in section 4.10. For general case, we will discuss in section 7.16)

The we will get the following consequence:

$$\text{If } M(\lambda)^i\neq 0 \text{ but } M(\lambda)^{i+1}=0, \text{ then } n=|\Phi^+_\lambda|.$$

pf. Set μ the anti-dominant weight corresponding to λ.
Then in right hand side of Jantzen sum formula, there are $|\Phi^+_\lambda|$ $\operatorname{ch}L(\mu)$'s and n $\operatorname{ch}L(\mu)$'s in the left side.

Exercise. Let λ be antidominant, regular and integral. In the Jantzen filtration of $M(w\cdot\lambda)$, prove that the number n above is just $l(w)$. so there are $l(w)+1$ layers in the filtration.

More general, if $\lambda\in\mathfrak{h}^*$ is antidominant and regular, while $w\in W_{[\lambda]}$, we have $|\Phi^+_{w\cdot\lambda}|=l_\lambda(w)$, where l_λ is the length function on $W_{[\lambda]}$ determined by $\Delta_{[\lambda]}$.

pf.
$$\begin{aligned}
n=|\Phi^+_{w\cdot\lambda}| &= |\{\alpha\in\Phi^+ : s_\alpha w\cdot\lambda<w\cdot\lambda\}| \\
&= |\{\alpha\in\Phi^+ : \langle w\cdot\lambda+\rho,\alpha^\vee\rangle\alpha>0\}| \\
&= |\{\alpha\in\Phi^+ : \langle\lambda+\rho,(w^{-1}\alpha)^\vee\rangle\in\mathbb{Z}^+\}| \\
&= |\{\alpha\in\Phi^+ : w^{-1}\alpha\in\Phi^-\}| \\
&= |\Phi^+\cap w(\Phi^-)| = l(w)
\end{aligned}$$

Some potential questions arising from Jantzen's theorem.

1) Is the filtration satisfying the conditions unique?
2) What is the multiplicities of those $M(\lambda)_i$
3) Are the filtration layers semisimple? If so, does the filtration coincide with one of standard module filtrations having semisimple quotients?
4) When $M(\mu)\hookrightarrow M(\lambda)$? More precisely, suppose $\mu\uparrow\lambda$ and set $r:=|\Phi^+_\lambda|-|\Phi^+_\mu|$. One might expect that $M(\mu)\subset M(\lambda)^i$ if $i\leq r$ while $M(\mu)\cap M(\lambda)^i=M(\mu)^{i-r}$ if $i\geq r$.

In particular, the hereditary property in (4) became known as the Jantzen Conjecture.

5.4 Example $\mathfrak{sl}_3(\mathbb{C})$

Recall: $\mathfrak{sl}_3$ has $\Delta=\{\alpha,\beta\}$, $\Phi^+=\{\alpha,\beta,\alpha+\beta\}$, $W=\{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha\}$

For anti-dominant, regular and integral λ. Consider $M(w\cdot\lambda)$

- $w=1$, $\quad L(\lambda)=M(\lambda)$
- $w=s_\alpha, s_\beta$. $\quad 0\to L(\lambda)\hookrightarrow M(w\cdot\lambda)\twoheadrightarrow L(w\cdot\lambda)\to 0$
- $w=s_\alpha s_\beta$. $\quad \{w'\in W: w'\cdot\lambda\leq w\cdot\lambda\}=\{1, s_\alpha, s_\beta\}$

$\Rightarrow \Phi^+_\lambda=\{\alpha, \alpha+\beta\}$

RHS of JSF $=\operatorname{ch} M(s_\alpha\cdot\lambda)+\operatorname{ch} M(s_\beta\cdot\lambda)$

On the other hand, $[M(w\cdot\lambda):L(\lambda)]=[M(w\cdot\lambda):L(w\cdot\lambda)]=1$.

Thus, $\sum_{i>0}\operatorname{ch} M(w\cdot\lambda)^i=\operatorname{ch} M(s_\alpha\cdot\lambda)+\operatorname{ch} M(s_\beta\cdot\lambda)$

$=2\operatorname{ch} L(\lambda)+\operatorname{ch} L(s_\alpha\cdot\lambda)+\operatorname{ch} L(s_\beta\cdot\lambda)$

Then $M(w\cdot\lambda)^i=0$, $\forall i>2$. and $M(w\cdot\lambda)^2=L(\lambda)$

$(2\cdot\operatorname{ch} M(\lambda)^2<\sum_{i>0}\operatorname{ch} M(w\cdot\lambda)^i)$

$\Rightarrow$ $M(w\cdot\lambda)$ has 4 composition factors and each has multi 1.

- $w=s_\beta s_\alpha$ $\quad$ Similar to the last case.
- $w=s_\alpha s_\beta s_\alpha$ $\quad \Phi^+_\lambda=\{\alpha,\beta,\alpha+\beta\}$

RHS of JSF $=\operatorname{ch} M(s_\beta s_\alpha\cdot\lambda)+\operatorname{ch} M(s_\alpha s_\beta\cdot\lambda)+\operatorname{ch} M(\lambda)$

$=3\operatorname{ch} L(\lambda)+2\operatorname{ch} L(s_\alpha\cdot\lambda)+2\operatorname{ch} L(s_\beta\cdot\lambda)$

$+\operatorname{ch} L(s_\alpha s_\beta\cdot\lambda)+\operatorname{ch} L(s_\beta s_\alpha\cdot\lambda)$

So $M(\lambda)^i=0$, $\forall i>3$, $\quad M(w\cdot\lambda)=L(\lambda)$

$M(w\cdot\lambda)^2$ has composition factors $L(s_\alpha\cdot\lambda)$, $L(s_\beta\cdot\lambda)$, $L(\lambda)$.

$M(w\cdot\lambda)^1=N(w\cdot\lambda)$ has composition factors $L(s_\alpha s_\beta\cdot\lambda)$ ⋯⋯ with multiplicity 1.

<u>Exercise</u>. Consider the principle block $\mathcal{O}_0$ for $\mathfrak{sl}_3$. It has 6 simple modules $L_w:=L(w\cdot(-2\rho))$ and corresponding Verma modules M_w. Set $\operatorname{ch} L_1=P$.

Compute $\mathrm{ch}\, L_w$ and show that all weight space have dimension 1.

Pf. $\quad s_\alpha \cdot 0 = -\langle \rho, \alpha^\vee \rangle \alpha = -\alpha, \quad s_\beta \cdot 0 = -\beta, \quad s_{\alpha+\beta} \cdot 0 = -2(\alpha+\beta) = -2\rho$

$s_\beta s_\alpha \cdot 0 = s_\beta \cdot (-\alpha) = -\alpha - 2\beta, \quad s_\alpha s_\beta \cdot 0 = -2\alpha - \beta$

$\Rightarrow \Phi_{[0]} = \{0, -\alpha, -\beta, -\alpha-2\beta, -2\alpha-\beta, -2\rho\}$

- $w = s_\alpha, \quad s_\alpha \cdot (-2\rho) = -\alpha - 2\beta$

$$\mathrm{ch}\, M(s_\alpha \cdot (-2\rho)) = \mathrm{ch}\, L_1 + \mathrm{ch}\, L_{s_\alpha}$$

On the other hand $\quad \mathrm{ch}\, M(-\alpha-2\beta) = \mathrm{ch}\, M(-2\rho) * e(\alpha)$

Note that $p = \tilde{f}_\alpha \tilde{f}_\beta \tilde{f}_{\alpha+\beta} = \prod_{\mu \in \Phi^+} (e(-\mu) + e(-2\mu) + \cdots)$

Thus $\quad \mathrm{ch}\, L_{s_\alpha} = p * e(\alpha) - p$

$$= (e(-\beta) + e(-2\beta) + \cdots)(e(-\alpha-\beta) + e(-2(\alpha+\beta)) + \cdots)$$

Since for each weight λ of L_{s_α}, there is only one integer combination of β and $\alpha+\beta$ expressing λ. Thus, coef of each $e(\mu)$ is 1 or 0.

- $w = s_\beta \quad$ Similar.

- $w = s_\beta s_\alpha, \quad w \cdot (-2\rho) = -\alpha$

$$\begin{aligned} \mathrm{ch}\, M_w &= \mathrm{ch}\, L_{s_\alpha} + \mathrm{ch}\, L_{s_\beta} + \mathrm{ch}\, L_1 + \mathrm{ch}\, L_w \\ &= (e(-\beta) + e(-2\beta) + \cdots)(e(-\alpha-\beta) + e(-2\alpha-2\beta) + \cdots) \\ &\quad + (e(-\alpha) + e(-2\alpha) + \cdots)(e(-\alpha-\beta) + e(-2\alpha-2\beta) + \cdots) \\ &\quad + (e(-\alpha) + e(-2\alpha) + \cdots)(e(-\beta) + e(-2\beta) + \cdots)(e(-\alpha-\beta) + e(-2\alpha-2\beta) + \cdots) \\ &\quad + \mathrm{ch}\, L_w \\ &= \tilde{f}_\beta \tilde{f}_{\alpha+\beta} + \tilde{f}_\alpha \tilde{f}_{\alpha+\beta} + \tilde{f}_\alpha \tilde{f}_\beta \tilde{f}_{\alpha+\beta} + \mathrm{ch}\, L_w \end{aligned}$$

On the other hand, $\qquad (1 = e(0))$

$$\begin{aligned} \mathrm{ch}\, M_w &= p * e(\alpha + 2\beta) \\ &= (1 + e(-\beta) + e(-2\beta) + \cdots)(e(-\alpha) + e(-2\alpha) + \cdots) \\ &\quad (1 + e(-\alpha-\beta) + e(-2\alpha-2\beta) + \cdots) \\ &= \tilde{f}_\alpha \tilde{f}_{\alpha+\beta} + \tilde{f}_\alpha + \tilde{f}_\beta \tilde{f}_\alpha + \tilde{f}_\beta \tilde{f}_\alpha \tilde{f}_{\alpha+\beta} \end{aligned}$$

$$\Rightarrow \mathrm{ch}\, L_w = \tilde{f}_\alpha + \tilde{f}_\beta \tilde{f}_\alpha - \hat{f}_\beta \hat{f}_{\alpha+\beta}$$

Note that for each weight λ, there is at most 1 integer combination of α and β expression, so is β and $\alpha+\beta$

Thus $\operatorname{ch} L_w = \sum_{0\le b<a} e(-a\alpha - b\beta) + \tilde{\mathcal{J}}_{\alpha+\beta}$

- $w = s_\alpha s_\beta$, $w\cdot(-2\rho) = -\beta$ Similar, we can get $\operatorname{ch} L_w = \sum_{0\le a<b} e(-a\alpha-b\beta) + \tilde{\mathcal{J}}_{\alpha+\beta}$
- $w = w_0$, $w\cdot(-2\rho) = 0$.

$$\begin{aligned}\operatorname{ch} M_{w_0} &= \operatorname{ch} L_{w_0} + \operatorname{ch} L_{s_\alpha} + \operatorname{ch} L_{s_\beta} + \operatorname{ch} L_{s_\alpha s_\beta} + \operatorname{ch} L_{s_\beta s_\alpha} + \operatorname{ch} L_1 \\ &= \operatorname{ch} L_{w_0} + \tilde{\mathcal{J}}_\beta \tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_\alpha \tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_\alpha + \tilde{\mathcal{J}}_\beta \tilde{\mathcal{J}}_\alpha - \tilde{\mathcal{J}}_\beta \tilde{\mathcal{J}}_{\alpha+\beta} \\ &\quad + \tilde{\mathcal{J}}_\beta + \tilde{\mathcal{J}}_\beta \tilde{\mathcal{J}}_\alpha - \tilde{\mathcal{J}}_\alpha \tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_\alpha \tilde{\mathcal{J}}_\beta \tilde{\mathcal{J}}_{\alpha+\beta}\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{ch} M_{w_0} &= p * e(\alpha+\beta+(\alpha+\beta)) \\ &= (1+\tilde{\mathcal{J}}_\alpha)(1+\tilde{\mathcal{J}}_\beta)(1+\tilde{\mathcal{J}}_{\alpha+\beta}) \\ &= e(0) + \tilde{\mathcal{J}}_\beta + \tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_{\alpha+\beta}\tilde{\mathcal{J}}_\beta + \tilde{\mathcal{J}}_\alpha + \tilde{\mathcal{J}}_\alpha\tilde{\mathcal{J}}_\beta + \tilde{\mathcal{J}}_\alpha\tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_\alpha\tilde{\mathcal{J}}_\beta\tilde{\mathcal{J}}_{\alpha+\beta}\end{aligned}$$

$$\Rightarrow \operatorname{ch} L_{w_0} = e(0) + \tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_\beta\tilde{\mathcal{J}}_{\alpha+\beta} + \tilde{\mathcal{J}}_\alpha\tilde{\mathcal{J}}_{\alpha+\beta} - \tilde{\mathcal{J}}_\alpha\tilde{\mathcal{J}}_\beta = e(0)$$

5.5 Application to BGG Theorem

Thm (BGG) If $[M(\lambda) : L(\mu)] \neq 0$, then μ is strongly linked to λ.

pf: We prove by induction on the number of linked weight $\mu \le \lambda$.

In the case, λ is minimal in the linkage class, i.e. $M(\lambda) = L(\lambda)$. Nothing to prove.

Suppose $[M(\lambda) : L(\mu)] \neq 0$ for $\mu < \lambda$. This means $[M', L(\mu)] \neq 0$.

JSF shows $[M(s_\alpha\cdot\lambda), L(\mu)] \neq 0$ for some $\alpha \in \Phi^+$.

Thus, by induction hyperthesis, $\mu \uparrow s_\alpha\cdot\lambda \uparrow \lambda \Rightarrow \mu \uparrow \lambda$.

5.6 Key Lemma

Let A be a PID (with unity),

- M : a free A-mod of rank r,
  $(\cdot,\cdot)$: a A-valued nondeg sym bilinear form of M.
  M^* : $\mathrm{Hom}_A(M,A)$ dual module (also has rank r)
  $M^\vee$: submod of M^* consisting of $e^\vee : f \mapsto (e,f)$ (also rank r)

By standard structure theory of PID, there is a basis $e_1^*, \dots, e_r^*$ of M^* s.t. $\{d_i e_i^*\}$ is the basis of $M^\vee$, where e_i^* are dual basis, $d_i \in A$.

(Actually we can require $d_1 | d_2 | \cdots | d_r$)

On the other hand, $\exists f_i$ a basis of M s.t. $f_i^\vee = d_i e_i^*$ (by defintion of $M^\vee$)

Fix two base of M, $(\cdot,\cdot)$ corresponds a matrix, which has nonzero det D.

Then $D = \prod_{i=1}^r d_i$ (up to units)

(transition matrix $\in GL(r,A)$, det = unit of A)

- Let p be a prime ($p|ab \Rightarrow p|a$ or $p|b$), for any $a \in A$, $\exists n \in \mathbb{N}$ s.t.

$$p^n | a \quad , \quad p^{n+1} \nmid a$$

prime ideal

Then define $v_p(a) = n$. **Prop** $v_p(ab) = v_p(a) + v_p(b)$

Note that $\overline{M} := M/pM$ is a vector space over a field $\overline{A} := A/pA$

For each $n \in \mathbb{Z}^+$, define $M(n) := \{e \in M : (e,M) \subseteq p^n A\}$ and $\overline{M(n)}$ to be the image under the canonical projection. $\quad \| \quad \{e \in M : e^\vee(M) \subseteq p^n A\}$

Obviously: $M = M(0) \supset M(1) \supset \cdots$

$\overline{M} = \overline{M(0)} \supset \overline{M(1)} \supset \cdots$ this behalf the filtration we desire

Lemma. a) $v_p(D) = \sum_{n>0} \dim_{\overline{A}} \overline{M(n)}$,

b) $\forall n$, the modified form $(e,f)_n := p^{-n}(e,f)$ on $M(n)$ induces a nondeg form on $\overline{M(n)}/\overline{M(n+1)}$

Pf. a) First, we consider $M(n)$.

Let $f = \sum a_i f_i \in M(n)$, where $a_i \in A$

$f \in M(n) \iff (\sum a_i f_i, M) \subseteq p^n A$.

$$\Leftrightarrow (\sum a_i f_i, e_j) \in p^n A \quad \forall j$$

$$\Leftrightarrow \sum_i a_i (f_i, e_j) \in p^n A \quad \forall j$$

$$\Leftrightarrow \sum a_i d_i e_i^*(e_j) \in p^n A \quad \forall j$$

$$\Leftrightarrow a_i d_i \in p^n A \quad \forall i$$

$$\Leftrightarrow v_p(a_i) + v_p(d_i) \geq n$$

Thus, $M(n) = \bigoplus_{i=1}^{r} \{a_i f_i : v_p(a_i) \geq n - v_p(d_i)\}$

Note that $\forall n > \max_i \{v_p(d_i)\}$, $\overline{M(n)} = 0$. Thus RHS is a fin sum.

If $0 \leq n \leq \max_i \{v_p(d_i)\}$, $\dim \overline{M(n)} = \#\{i \mid v_p(a_i) \geq n - v_p(d_i)\}$. Precisely, $\overline{M(n)}$ has a basis $\{[f_i] \mid v_p(d_i) \geq n\}$, similar for $\{[e_i] \mid v_p(d_i) \geq n\}$ ← representives

$$\Rightarrow \sum_{i>0} \dim \overline{M(n)} = \sum_{j>0} \#\{i \mid v_p(d_i) \geq j\} = \sum_{i=1}^{r} \sum_{j=1}^{v_p(d_i)} 1 = \sum v_p(d_i)$$

$$= v_p(\prod d_i) = v_p(D)$$

Rmk: Here we know $\{\overline{f_i} : v_p(d_i) = n\}$ & $\{\overline{e_i} : v_p(d_i) = n\}$ are two base of $\overline{M(n)}/\overline{M(n+1)}$

b) $\forall \overline{e}, \overline{f} \in \overline{M(n)}$, where $e, f \in M(n)$. Define

$$(\overline{e} + \overline{M(n+1)}, \overline{f} + \overline{M(n+1)})_n := \overline{(e,f)_n}$$

(M/pM, A/pA)

• well-def: $\forall e', f' \in M(n+1)$,

$$(\overline{e} + \overline{e'} + \overline{M(n+1)}, \overline{f} + \overline{f'} + \overline{M(n+1)})_n = \overline{(e+e', f+f')_n}$$

$$= \overline{(e,f)_n} + \overline{p^{n+1} \cdot p^{-n} \cdot \text{const}} = \overline{(e,f)_n}$$

$$\forall e', f' \in pM, \ (\overline{e+e'} + \overline{M(n+1)}, \overline{f+f'} + \overline{M(n+1)})_n = \cdots = \overline{(e,f)_n}$$

• nondeg: Now we write $(\overline{e}, \overline{f})_n$ in short.

In (a), we prove that $\{e_i \mid v_p(d_i) \geq n\}$ & $\{f_i \mid v_p(d_i) \geq n\}$ are two base of $M(n)$.

Note that $(\overline{e_i}, \overline{f_j})_n = \overline{(e_i, f_j)_n} = \delta_{ij} \cdot \overline{d_i p^{-n}}$. Thus

$(\overline{e_i}, \overline{f_j})_n \neq 0$ iff $i = j$ & $v_p(d_i) = n$, i.e. the matrix is $\begin{pmatrix} \ddots & & \\ & \overline{d_i p^{-n}} & \\ & & \ddots \end{pmatrix}$

Therefore, the form is nondegerated.

5.7 Proof of Jantzen's theorem

Notation: $K = \mathbb{C}(T)$, $A = \mathbb{C}[T]$, PID where T is an indeterminate

$$\mathfrak{g}_K = K \otimes_{\mathbb{C}} \mathfrak{g}, \quad \mathfrak{g}_A = A \otimes_{\mathbb{C}} \mathfrak{g}, \quad \mathfrak{h}_K^* = \mathrm{Hom}_K(\mathfrak{h}_K, K)$$

$\forall \lambda \in \mathfrak{h}^*$, denote $\lambda_T := \lambda + T\rho \in \mathfrak{h}_K^*$.

$M(\lambda_T) := \mathcal{U}(\mathfrak{g}_K) \otimes_{\mathcal{U}(\mathfrak{b}_K)} \mathbb{C}_{\lambda_T}$, the Verma module of $\mathfrak{g}_K$.

$M(\lambda_T)_A := \mathcal{U}(\mathfrak{g}_A) \otimes_{\mathcal{U}(\mathfrak{b}_A)} \mathbb{C}_{\lambda_T}$, a $\mathfrak{g}_A$-submod of $M(\lambda_T)$

$M_\mu := (M(\lambda_T)_\mu)_A$, where $\mu \in \Pi(M(\lambda_T)) = \lambda_T - \Gamma$

Consequences: • λ_T is anti-dominant: $\langle \lambda_T + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Phi$

- $M(\lambda_T)$ has a nondeg contravariant form: $M(\lambda_T)$ simple & Thm 3.15
- By construction, the restriction of the contra. form to $M_{\lambda_T - \nu}$ is A-valued nondeg sym bilinear, denoted by $(\cdot, \cdot)$.
- Each $M_{\lambda_T-\nu}$ has finite rank as free A-mod.
- By previous lemma, taking prime $T \in A$, there are descending chains of each weight space $M_{\lambda_T - \nu}$ with fin length.

$$M_{\lambda_T-\nu} \supset M_{\lambda_T-\nu}(1) \supset M_{\lambda_T-\nu}(2) \supset \cdots$$

Recall $M_{\lambda_T-\nu}(i) = \{ e \in M_{\lambda_T-\nu} : (e, M_{\lambda_T-\nu}) \subseteq T^i A \}$

Denote $M(\lambda_T)_A^i := \sum_{\nu \in \Gamma} M_{\lambda_T-\nu}(i)$

Thm (Jantzen). Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda)$ has a filtration by submodules:

$$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \cdots$$

with $M(\lambda)^i = 0$ for large enough i, satisfying:

a) Each nonzero quotient has a nondegenerate contravariant form.
b) $M^1(\lambda) = N(\lambda)$, the unique maximal submodule of $M(\lambda)$
c) The formal characters satisfy

$$\sum_{i>0} \mathrm{ch}(M(\lambda)^i) = \sum_{\alpha>0,\ s_\alpha \cdot \lambda < \lambda} \mathrm{ch}\, M(s_\alpha \cdot \lambda)$$

Pf. Step 0. Construct the filtration

Claim that: $M(\lambda) \cong \overline{M(\lambda_T)^0_A} \supset \overline{M(\lambda_T)^1_A} \supset \cdots$ is a fin filtration of submods

- $M(\lambda) \cong \overline{M(\lambda_T)^0_A}$:

$M(\lambda_T)_A / TM(\lambda_T)_A$ is a $A/TA \cong \mathbb{C}$-linear space with a natural $\mathfrak{g}$-action. Let $\varphi: M(\lambda) \to M(\lambda_T)_A / TM(\lambda_T)_A$ via $\varphi(v_\lambda) = [v_{\lambda_T}]$

Obviously, φ is a $\mathfrak{g}$-homo. and bilinear $\quad \dfrac{\mathcal{U}(\mathfrak{n}_A^-) \otimes_A \mathbb{C}}{T\,\mathcal{U}(\mathfrak{n}_A^-) \otimes_A \mathbb{C}} \cong \mathcal{U}(\mathfrak{n}^-)$

- $\overline{M(\lambda_T)^i_A}$ is $\mathfrak{g}$-mod:

$TM(\lambda_T)_A$ is a $\mathfrak{g}$-mod, thus $\overline{M(\lambda_T)^i_A}$ is also a $\mathfrak{g}$-mod.

- Finite filtration:

For each linked weight $\lambda - \nu$ of λ and $\lambda - \nu < \lambda$.
$M_{\lambda_T - \nu}(i) = 0$ for sufficiently large i. Then $\dim(\overline{M(\lambda_T)^i_A})_{\lambda-\nu} = 0$ for large i
Thus it is a finite filtration.

Step 1. Prove (a) & (b)

Now we set $M(\lambda) \supset M^1 \supset M^2 \supset \cdots$ being the corresponding chain for $M(\lambda)$

restriction from the form of $M(\lambda_T)$

By previous lemma, the modified form of $M(\lambda_T)^i_A$ induces a nondeg form of $\overline{M(\lambda_T)^i_A} / \overline{M(\lambda_T)^{i+1}_A}$. Since the form of $M(\lambda_T)$ is contra. the induced form is also contra. (One can check directly)

Since $M(\lambda)/M^1$ has nondeg contra. form, by thm 3.15, it is simple.
Thus, M^1 is the maximal submod and $M(\lambda)/M^1 \cong L(\lambda)$.

Step 2. JSF: $\sum_{i>0} \mathrm{ch}\, M^i = \sum_{\alpha \in \Phi^+_\lambda} \mathrm{ch}\, M(s_\alpha \cdot \lambda)$,

Here we need a proposition which will be proved in the following sections:

The determinant of nondeg contra. form of $M(\lambda_T)_{\lambda_T - \nu}$ is
$D_\nu(\lambda_T) = \prod_{\alpha>0} \prod_{r>0} (\langle \lambda_T + \rho, \alpha^\vee \rangle - r)^{P(\nu - r\alpha)}$, where P is the Kostant partition function.

Then $\text{LHS} = \sum_{\nu\in\Gamma}\sum_{i>0} \dim_{\mathbb{C}} M^i_{\lambda-\nu} \cdot e(\lambda-\nu)$ $\quad$ $(\cdot,\cdot)$ has the same matrices on $M(\lambda_T)_A$ & $M(\lambda_T)$

$$= \sum_{\nu\in\Gamma}\sum_{i>0} \dim_{\mathbb{C}} (M(\lambda_T)^i_A)_{\lambda_T-\nu} \cdot e(\lambda-\nu)$$

$$= \sum_{\nu\in\Gamma} \nu_T(D_\nu(\lambda_T)) \cdot e(\lambda-\nu)$$

$$= \sum_{\nu\in\Gamma}\sum_{\alpha>0}\sum_{r>0} P(\nu-r\alpha)\, \nu_T(\langle\lambda_T+\rho,\alpha^\vee\rangle - r)\cdot e(\lambda-\nu)$$

Note that $\langle\lambda_T+\rho,\alpha^\vee\rangle - r = \langle\lambda+\rho,\alpha^\vee\rangle - r + T\langle\rho,\alpha^\vee\rangle$. Thus

$$\nu_T(\langle\lambda_T+\rho,\alpha^\vee\rangle - r) = 1 \text{ if } \langle\lambda+\rho,\alpha^\vee\rangle = r \Leftrightarrow \alpha\in\Phi^+_\lambda, \text{ otherwise } 0.$$

Thus, $\text{LHS} = \sum_{\nu\in\Gamma}\sum_{\alpha\in\Phi^+_\lambda} P(\nu-\langle\lambda+\rho,\alpha^\vee\rangle\alpha)\cdot e(\lambda-\nu)$

$$= \sum_{\alpha\in\Phi^+_\lambda}\sum_{\nu\in\Gamma} P(\nu)\, e(\lambda-\langle\lambda+\rho,\alpha^\vee\rangle\alpha-\nu)$$

$$= \sum_{\alpha\in\Phi^+_\lambda} \operatorname{ch} M(s_\alpha\cdot\lambda)$$

5.8 Determinant Formula

Recall: In 3.15, we defined universal bilinear form $C: \mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ via

$$C(u,u') = \varphi(\tau(u)u') = \varepsilon^-\otimes \mathrm{id}\otimes\varepsilon^+(\tau(u)u')$$

If $\nu\in\Gamma$, we write C_ν the restriction to fin dim wt space $\mathcal{U}(\mathfrak{n}^-)_{-\nu}$. Relative an ordered basis, C_ν gives Shapovalov matrix S_ν, which is sym and has determinant D_ν.

Thm. Fix $\nu\in\Gamma$. $D_\nu\neq 0$ & up to a nonzero factor,

$$D_\nu = \prod_{\alpha>0}\prod_{r>0}\left(h_\alpha + \langle\rho,\alpha^\vee\rangle - r\right)^{P(\nu-r\alpha)}$$

<u>Exercise</u>. $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $\nu = \alpha+\beta$. Relative to the ordered basis $\{y_\alpha y_\beta, y_\gamma\}$ of $\mathcal{U}(\mathfrak{n}^-)_{-\nu}$ (recalling $h_\gamma = h_\alpha + h_\beta$),

$$S_\nu = \begin{pmatrix} h_\alpha h_\beta + h_\beta & -h_\beta \\ -h_\beta & h_\alpha + h_\beta \end{pmatrix}$$

Check $\det S_\nu = h_\alpha h_\beta(h_\alpha+h_\beta+1)$, in agreement with the theorem.

Pf. $(y_\alpha y_\beta, y_\alpha y_\beta) = \varphi(x_\beta x_\alpha y_\alpha y_\beta)$

$$= \varphi(x_\beta h_\alpha y_\beta + y_\alpha x_\beta y_\beta x_\alpha)$$

$$= \varphi(x_\beta y_\beta h_\alpha + x_\beta y_\beta + y_\alpha h_\beta x_\alpha)$$

$$= \varphi(y_\beta x_\beta h_\alpha) + h_\beta h_\alpha + h_\beta$$

$$= h_\beta h_\alpha + h_\beta$$

$(y_\alpha y_\beta, y_\gamma) = \varphi(x_\beta x_\alpha y_{\alpha+\beta})$

$$= \varphi(x_\beta y_{\alpha+\beta} x_\alpha + x_\beta y_\beta)$$

$$= \varphi(y_{\alpha+\beta} x_\beta x_\alpha + y_\alpha x_\alpha - h_\beta + y_\beta x_\beta)$$

$$= -h_\beta$$

$(y_\gamma, y_\gamma) = \varphi(x_\gamma y_\gamma) = \varphi(h_\gamma + y_\gamma x_\gamma) = h_\gamma$

By the formula, $D_{\alpha+\beta} = \prod_{\mu>0} \prod_{r>0} (h_\mu + \langle \rho, \mu^\vee \rangle \mu - r)^{P(\alpha+\beta - r\mu)}$

$$= (h_\alpha + \langle \rho, \alpha^\vee \rangle - 1)^{P(\beta)} (h_\beta + \langle \rho, \beta^\vee \rangle - 1)^{P(\alpha)}$$

$$(h_{\alpha+\beta} + \langle \rho, \gamma^\vee \rangle - 1)^{P(0)}$$

$$= h_\alpha h_\beta (h_\alpha + h_\beta + 1)$$